<u>dem Th6</u> On suppose $f : E \to F$ linéaire.

On se donne $\mathcal{B} = (\vec{e_1}, \dots, \vec{e_p})$ une base de E

et $\mathcal{E} = (\vec{\varepsilon_1}, \dots, \vec{\varepsilon_n})$ une base de F.

On se donne aussi $\vec{x} \in E$ et $\vec{y} \in F$.

On note
$$A \overset{\text{def}}{=} \mathrm{Mat}(f ; \mathcal{B}, \mathcal{E}) = ((a_{i,j}))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$
$$X = \mathrm{Mat}(\vec{x}, \mathcal{B}) = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$$
$$Y = \mathrm{Mat}(\vec{y} ; \mathcal{E}) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

<u>but :</u> $\vec{y} = f(\vec{x}) \iff Y = A \times X$

On a : $\vec{x} \overset{\text{def}}{=} \sum_{j=1}^{p} x_j \cdot \vec{e_j}$ et $\vec{y} \overset{\text{def}}{=} \sum_{i=1}^{n} y_i \cdot \vec{\varepsilon_i}$

Donc : $f(\vec{x}) = f\left(\sum_{j=1}^{p} x_j \cdot \vec{e_j}\right) = \sum_{j=1}^{p} x_j \cdot f(\vec{e_j})$

Mais pour tout $j \in [\![1, p]\!]$:
$$f(\vec{e_j}) \overset{\text{def}}{=} \sum_{i=1}^{n} a_{i,j} \cdot \vec{\varepsilon_i}$$

Ainsi
$$f(\vec{x}) = \sum_{j=1}^{p} x_j \cdot \left(\sum_{i=1}^{n} a_{i,j} \cdot \vec{\varepsilon_i}\right) = \sum_{j=1}^{p} \left(\sum_{i=1}^{n} a_{i,j} \times x_j \cdot \vec{\varepsilon_i}\right)$$
$$\underset{\text{Fubini}}{=} \sum_{i=1}^{n} \left(\sum_{j=1}^{p} a_{i,j} \times x_j \cdot \vec{\varepsilon_i}\right)$$
$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{p} a_{i,j} \times x_j\right) \cdot \vec{\varepsilon_i}$$

Par unicité des coordonnées dans la base $\mathcal{E} = (\vec{\varepsilon_1}, \dots, \vec{\varepsilon_n})$:

$$\vec{y} = f(\vec{x}) \iff \forall i \in [\![1, n]\!], \quad y_i = \sum_{j=1}^{p} a_{i,j} \times x_j$$

$$\iff Y = A \times X$$

<u>Rem:</u> On a donc

$$\mathrm{Mat}(f(\vec{x}), \mathcal{E}) = \mathrm{Mat}(f, \mathcal{B}, \mathcal{E}) \times \mathrm{Mat}(\vec{x}, \mathcal{B})$$