3. Analyse Soit $P_n \in \mathbb{K}[X]$ tel que $P_n - P_n' = X^n$

On a donc $P_n \neq 0_{\mathbb{K}[X]}$ et donc $\deg(P_n') < \deg P_n$

donc $\deg(P_n - P_n') = \deg P_n$

Comme $P_n - P_n' = X^n$ on a donc $\deg P_n = n$

On note $P_n = \sum_{k=0}^{n} a_k X^k$ avec $(a_0, \dots, a_n) \in \mathbb{K}^{n+1}$ et $a_n \neq 0$

Alors $P_n' = \sum_{k=1}^{n} k a_k X^{k-1} = \sum_{k=0}^{n-1} (k+1) a_{k+1} X^k$

Donc $P_n - P_n' = X^n \iff a_n X^n + \sum_{k=0}^{n-1} (a_k - (k+1)a_{k+1}) X^k = X^n$

$$\iff \begin{cases} a_n = 1 \\ \forall k \in [\![0, n-1]\!], \quad a_k = (k+1) a_{k+1} \end{cases}$$

$$\iff \begin{cases} a_n = 1 = \frac{n!}{n!} \\ a_{n-1} = n = \frac{n!}{(n-1)!} \\ a_{n-2} = n(n-1) = \frac{n!}{(n-2)!} \\ \vdots \\ a_0 = n(n-1) \times \dots \times 2 \times 1 = n! = \frac{n!}{0!} \end{cases}$$

$$\iff \forall k \in [\![0, n]\!], \quad a_k = \frac{n!}{k!}$$

Donc $P_n = \sum_{k=0}^{n} \frac{n!}{k!} X^k$. Ainsi si il existe alors P_n est unique.

<u>Synthèse</u> On pose $P_n = \sum_{k=0}^{n} \frac{n!}{k!} X^k$

Alors $P_n' = \sum_{k=1}^{n} \frac{n!}{k!} k X^{k-1} = \sum_{k=1}^{n} \frac{n!}{(k-1)!} X^{k-1} = \sum_{k=0}^{n-1} \frac{n!}{k!} X^k$

donc $P_n - P_n' = \frac{n!}{n!} X^n = X^n$.

<u>Conclusion</u>

$$\boxed{\begin{array}{l} \exists !\, P_n \in \mathbb{K}[X],\ P_n - P_n' = X^n. \\ \text{De plus } P_n = n! \times \sum_{k=0}^{n} \frac{X^k}{k!} \end{array}}$$